Resit exam Analysis on Manifolds

Tuesday, July 2, 9:00–12:00

This exam consists of four assignments. You get 10 points for free. Using notes or books is not allowed. It is not necessary to write the questions on your answer sheet.

Problem 1. We identify the space $M(2,\mathbb{R})$ of real 2×2-matrices with $\mathbb{R}^4$, by associating the matrix

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

with the point $(x_{11}, x_{12}, x_{21}, x_{22}) \in \mathbb{R}^4$. Let I be the identity matrix, i.e.,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Consider the set $O(2,\mathbb{R})$ of orthogonal matrices, given by

$$O(2,\mathbb{R}) = \{A \in M(2,\mathbb{R}) \mid A^\top A = I\}.$$

(a) (10 points.) Show that the map $g\colon (0,2\pi) \to M(2,\mathbb{R})$ given by

$$g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

satisfies $g(\phi) \in O(2,\mathbb{R})$ for any $\phi \in (0,2\pi)$ and that it is a local parametrization for $O(2,\mathbb{R})$.

(b) (5 points.) Show that the map ${}^{-1}\colon O(2,\mathbb{R}) \to O(2,\mathbb{R})$ that sends a matrix $A \in O(2,\mathbb{R})$ to its inverse A^{-1} is smooth.

Problem 2. (10 points.) Take an $r \in \mathbb{R}$ and set $F(x,y,z) = x^2 + y^2 - z^2 - r$. For what values of $r \in \mathbb{R}$ does the equation $F(x,y,z) = 0$ define a submanifold of $\mathbb{R}^3$?

Problem 3. Recall that an n-dimensional smooth manifold M is said to be *orientable* if there exists a smooth atlas $\mathcal{A}$ covering M such that the determinant of the transition map of any two charts from $\mathcal{A}$ has positive determinant; i.e. if (U,φ) and (V,ψ) are two charts from $\mathcal{A}$ with $U \cap V \neq \varnothing$ and $\kappa := \psi \circ \varphi^{-1}$ then

$$\det(D\kappa) > 0 \quad \text{on } U \cap V.$$

Suppose that M is such that the tangent bundle TM of M is of the form $M \times \mathbb{R}^k$. This implies that there exists a set of n vector fields $V_1, \ldots, V_n$ on M such that the n tangent vectors $V_i(p) = (V_i)_p \in T_pM$ at $p \in M$ are linearly independent for all p (you do not have to show this).

(a) (5 points.) In any coordinate chart (U, φ) with coordinates x^i the vector field V_j may be written as

$$V_j = \sum_{i=1}^{n} c_j^i \frac{\partial}{\partial x^i}$$

for some set of functions c_j^i. Show that the determinant of the matrix $\left(c_j^i(p)\right)$ is nonzero and has the same sign for all $p \in U$.

(b) (5 points.) If $\det\left(c_j^i(p)\right)$ is negative on the subset U, construct a new coordinate chart $(U, \widetilde{\varphi})$ on the same subset U such that with respect to the new coordinate chart the determinant is positive.

(c) (10 points.) Take a second coordinate chart (V, ψ) having non-empty overlap with U and coordinates y^j. We may write

$$V_j = \sum_{k=1}^{n} b_j^k \frac{\partial}{\partial y^k}$$

for some set of functions b_j^k, and, after modifying the coordinate chart as in (b) if necessary, we may assume that $\det\left(b_j^k(p)\right) > 0$ on V. Prove that the coefficients b_j^k and c_j^i are related to each other by

$$b_j^k = \sum_{i=1}^{n} c_j^i \frac{\partial y^k}{\partial x^i}.$$

(d) (10 points.) Prove that M is orientable.

Problem 4. On $\mathbb{R}^3 \setminus \{O\}$ the 2-form ω is given by

$$\omega = \frac{x\, \mathrm{d}y \wedge \mathrm{d}z + y\, \mathrm{d}z \wedge \mathrm{d}x + z\, \mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}. \quad (1)$$

Here $O = (0, 0, 0)$.

(a) (8 points.) Prove that ω is closed.

S_ε is defined to be the surface of the sphere with center O and radius $\varepsilon > 0$, i.e., $S_\varepsilon = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = \varepsilon^2\}$.

(b) (7 points.) Compute $\int_{S_\varepsilon} \omega$. (Hint: note that the denominator in the right hand side of (1) is constant on S_ε.)

Let D be a 3-dimensional submanifold of $\mathbb{R}^3$ with nonempty boundary $\partial D =: M$.

(c) (10 points.) Show that $\int_M \omega = 0$ if $O \notin D$.

(d) (10 points.) Show that $\int_M \omega = 4\pi$ if $O \in D$. (Hint: for sufficiently small $\varepsilon > 0$ the sphere S_ε lies entirely within D. Consider the manifold with boundary formed by the region between S_ε and M.)